

QCD Calculations of Heavy Quarkonium States *

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Recent results on the QCD analysis of bound states of heavy $\bar{q}q$ quarks are reviewed, paying attention to what can be derived from the theory with a reasonable degree of rigour. We report a calculation of $\bar{b}c$ bound states; a very precise evaluation of b, c quark masses from quarkonium spectrum; the NNLO evaluation of $\Upsilon \rightarrow e^+e^-$; and a discussion of power corrections. For the b quark *pole* mass we get, including $O(m_c^2/m_b^2)$ and $O(\alpha_s^5 \log \alpha_s)$ corrections, $m_b = 5.020 \pm 0.058$ GeV; and for the \overline{MS} mass the result, correct to $O(\alpha_s^3)$, $O(m_c^2/m_b^2)$, $\bar{m}_b(\bar{m}_b) = 4.286 \pm 0.036$ GeV. For the decay $\Upsilon \rightarrow e^+e^-$, higher corrections are too large to permit a reliable calculation, but we can predict a toponium width of 13 ± 1 keV.

1. INTRODUCTION

In the present note we are going to review some aspects of the QCD analysis of heavy quarkonia, $t\bar{t}$, $c\bar{c}$, $b\bar{c}$ and especially $b\bar{b}$ states. The validity of *ab initio* QCD calculations varies from one case to another. For the energy levels of toponium with $n \leq 5$, n being the principal quantum number, and for the energy of the ground state in bottomium, we have a very satisfactory situation, almost comparable to that in positronium calculations. For bottomium hyperfine splitting, and for the energy of the *charmonium* ground state, the situation is less favourable, although reasonably under control. Less favourable still is what one has for the energy levels of bottomium with $n = 2$. Here NLO and NNLO perturbative and the leading nonperturbative corrections are comparable to the LO results. The same is true for the decay $\Upsilon \rightarrow e^+e^-$, and even for toponium decay, $T \rightarrow e^+e^-$, higher corrections are a bit too large for comfort. For higher states, for the decays of $n = 2$ bottomium or for the ground state of charmonium, a *rigorous* QCD evaluation is out of the question. There are methods that have been devised in the literature to deal with this, essentially more or less justified models; we do not discuss them now. The interested reader may find information, and references, in e.g. my Lisbon lectures^[1].

This does not mean that even in the favourable cases all questions are settled, and in particular we discuss here power-like corrections, *at short distances*, where there is still some controversy.

2. NONRELATIVISTIC QCD POTENTIAL, AND CORRECTIONS

To analyze the lowest states of heavy quarks one proceeds as follows: first, in the nonrelativistic (NR) approximation one finds the potential that governs the dynamics. This potential is evaluated to increasing orders of perturbation theory; at present it is known to one and two^[2] loops. One can thus write

$$H = H^{(0)} + H_{\log}; \quad (2.1a)$$

where $H^{(0)}$ may be solved exactly and contains the coulombic part of the interaction. For $\bar{q}q$ states,

$$H^{(0)} = 2m + \frac{-1}{m} \Delta - \frac{C_F \tilde{\alpha}_s(\mu^2)}{r}, \quad (2.1b)$$

$$\tilde{\alpha}_s(\mu^2) = \alpha_s(\mu^2) \left\{ 1 + c^{(1)} \frac{\alpha_s(\mu^2)}{\pi} + c^{(2)} \frac{\alpha_s^2(\mu^2)}{\pi^2} \right\};$$

$$c^{(1)} = a_1 + \frac{\gamma_E \beta_0}{2}$$

$$c^{(2)} = \gamma_E \left(a_1 \beta_0 + \frac{\beta_1}{8} \right) + \left(\frac{\pi^2}{12} + \gamma_E^2 \right) \frac{\beta_0}{4} + a_2.$$

The expression for the a_i may be found in e.g. ref. 1, m is the pole mass of the quark, and H_{\log} is

$$H_{\log} = \frac{-C_F \beta_0 \alpha_s(\mu^2)^2 \log r \mu}{2\pi} \frac{1}{r} + \frac{-C_F \beta_0^2 \alpha_s^3 \log^2 r \mu}{4\pi^2} \frac{1}{r} + \frac{-C_F \alpha_s^3}{\pi^2} \left(a_1 \beta_0 + \frac{\beta_1}{8} + \frac{\gamma_E \beta_0^2}{2} \right) \frac{\log r \mu}{r}.$$

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One solves exactly the coulombic Schrödinger equation,

$$H^{(0)}\Psi_{nl} = E_{nl}^{(0)}\Psi_{nl}, \quad (2.2)$$

treating the terms in H_{\log} as perturbations. Relativistic corrections are then added, to be also treated as perturbations. These relativistic corrections include spin-dependent ones, known at tree level since a long time (they are essentially like for positronium) and to one loop from refs. 3,4. Moreover, and because of the nonabelian character of QCD, we have mixed relativistic–one loop corrections for the spin-independent piece, first evaluated in ref. 5 using the method of equivalent potentials of Gupta et al. (see e.g. ref. 4). Concentrating on the spin-independent hamiltonian we then write

$$H_{\text{SI}}^{\text{e.p.}} = H^{(0)} + H_{\log} + \frac{-1}{4m^3}\Delta^2 + \frac{C_F\alpha_s}{m^2r}\Delta + \frac{C_F b_1 \alpha_s^2}{2mr^2} \quad (2.3)$$

and $b_1 = \frac{1}{2}C_F - 2C_A$. Here the superindex “e.p.” means that the hamiltonian was obtained with the method of equivalent potentials.

The hamiltonian has been re-evaluated recently by Brambilla et al.^[6]; these authors generalize the nonrelativistic (heavy quark) effective theory (NRQCD^[7]), matching then the Green’s functions to a potential description (pNRQCD^[8]). In this way they find,

$$\begin{aligned} H_{\text{SI}}^{\text{pHQET}} = & H^{(0)} + H_{\log} + \frac{-1}{4m^3}\Delta^2 \\ & + \frac{\pi C_F \alpha_s}{m^2}\delta(\mathbf{r}) + \frac{C_F \alpha_s}{2r^3 m^2}\mathbf{L}^2 - \frac{C_F \alpha_s}{2m^2}\left\{\frac{1}{r}, \Delta\right\} \\ & + \frac{-C_F C_A \alpha_s^2}{2mr^2} \end{aligned} \quad (2.4)$$

This is *different* from (2.3). The difference is due to the fact that in the derivation of (2.3) one works with the S-matrix, while for (2.4) one uses Green’s functions. The difference between (2.3) and (2.4) vanishes when taking expectation values between coulombic wave functions, say, solutions of (2.2): so they will produce the same energy spectrum, at least to order α_s^4 . At higher orders, (2.3) and (2.4) would produce different results; but also new terms will appear in the two formalisms: their equivalence, or non-equivalence, has not been proved. For numerical results to $O(\alpha_s^4)$ for all states see ref. 9; here we present those

for the ground state energy. We write

$$E_{10}^{\text{p.t.}} = 2m - m \frac{C_F^2 \tilde{\alpha}_s^2}{4} + \delta_{\text{p.t.}} E_{10}. \quad (2.5a)$$

The label “p.t.” in $E_{nl}^{\text{p.t.}}$ indicates that we have as yet only used results deduced from perturbation theory; the full expression would be

$$E_{nl} = E_{nl}^{\text{p.t.}} + \delta_{\text{NP}} E_{nl}.$$

$\delta_{\text{NP}} E_{nl}$ embodies the nonperturbative contributions, to be discussed below. The $\delta_{\text{p.t.}} E_{10}$ is, with $a = 2/mC_F \tilde{\alpha}_s$,

$$\begin{aligned} \delta_{\text{p.t.}} E_{10} = & -\frac{10}{8m^3 a^4} - \frac{3C_F \alpha_s}{m^2 a^3} + \frac{C_F b_1 \alpha_s^2}{m a^2} \\ & - \frac{C_F c_2^{(L)} \alpha_s^3}{\pi^2 a} \left[\log \frac{a\mu}{2} + \psi(2) \right] \\ & - \frac{C_F \beta_0^2 \alpha_s^3}{4\pi^2 a} \left\{ \log^2 \frac{a\mu}{2} - \frac{\gamma_E}{2} \log \frac{a\mu}{2} \right. \\ & \left. + \frac{3 + 3\gamma_E^2 - \pi^2 + 6\zeta_3}{12} \right\}, \end{aligned} \quad (2.5b)$$

$$c_2^{(L)} = a_1 \beta_0 + \frac{1}{8} \beta_1 + \frac{1}{2} \gamma_E \beta_0^2.$$

The (nominally) leading, $\log \alpha_s$ corrections of next order are also known. They include a logarithmic correction to the static potential^[10], and a relativistic one-loop correction^[5,11]. With the full result (as given in ref. 6) one has, for $\mu = 2/a$,

$$\delta_{[\alpha_s^5 \log \alpha_s]} E_{10} = -m [C_F + \frac{3}{2}C_A] C_F^4 \alpha_s^5 (\log \alpha_s) / \pi. \quad (2.5c)$$

In the calculations we will include, for the b quark case, a correction^[11] of order m_c^2/m_b^2 ,

$$\delta_{[m_c^2/m_b^2]} E_{10} = 2m_b \frac{3T_F \alpha_s}{2\pi} \frac{m_c^2}{m_b^2}. \quad (2.5d)$$

We can invert (2.5) to obtain a precise determination of c , b quark masses from those of the J/ψ , Υ particles. The determination includes nonperturbative effects, to be discussed below. As input parameters we take the recent determination^[12],

$$\begin{aligned} A(n_f = 4, \text{ three loops}) &= 0.283 \pm 0.035 \text{ GeV} \\ &[\alpha_s(M_Z^2) \simeq 0.117 \pm 0.024], \end{aligned}$$

and for the gluon condensate the value $\langle \alpha_s G^2 \rangle = 0.06 \pm 0.02 \text{ GeV}^4$.

Then comes the matter of the renormalization point, μ . As Eqs. (2.5b,c) show, logarithms are avoided until order α_s^5 if choosing $\mu = \mu_0 = 2/a$,

and this will be our central value here. Then, for the b quark, we find

$$\begin{aligned}
m_b &= 5.020 \pm 0.043 (\Lambda) \mp 0.005 (\langle \alpha_s G^2 \rangle) \\
&\quad {}_{-0.037}^{+0.031} \text{ (vary } \mu^2 \text{ by 25\%)} \\
&\quad \pm 0.006 \pm 0.017 \text{ (other th. uncert.)} \\
&= 5.020 \pm 0.058 \text{ GeV}.
\end{aligned} \tag{2.6a}$$

To obtain this, we have included the correction of Eq. (2.5c), and for the “theoretical error” also that estimated in ref. 13 for the perturbative evaluations. The value of the $\overline{\text{MS}}$ mass that corresponds to this is, taking into account $O(\alpha_s^3)$ and leading $O(m_c^2/m_b^2)$ corrections^[14],

$$\bar{m}_b(\bar{m}_b) = 4272 \pm 43 \text{ MeV}. \tag{2.6b}$$

We present a summary of QCD calculations of the quark masses, with increasing accuracy.

Ref.	$m_b(\text{pole})$	$\bar{m}_b(\bar{m}_b^2)$	$m_c(\text{pole})$
TY	4971 ± 72	4401_{-35}^{+21}	$1585 \pm 20^*$
PY	5065 ± 60	4455_{-29}^{+45}	1866_{-133}^{+215}
Here	5022 ± 58	4272 ± 43	—

b, c quark masses. (*) Systematic errors not included.

TY: Titard and Ynduráin^[5]. $O(\alpha_s^3)$ plus $O(\alpha_s^3)v$, $O(v^2)$ for m ; $O(\alpha_s^2)$ for \bar{m} . Rescaled for $\Lambda(n_f = 4) = 283 \text{ MeV}$.

PY: Pineda and Ynduráin^[9]. Full $O(\alpha_s^4)$ for m ; $O(\alpha_s^2)$ for \bar{m} . Rescaled for $\Lambda(n_f = 4) = 283 \text{ MeV}$.

Here: This calculation. $O(\alpha_s^4)$, $O(\alpha_s m_c^2/m_b^2)$ and $O(\alpha_s^5 \log \alpha_s)$ for m ; $O(\alpha_s^3)$ and $O(\alpha_s^2 m_c^2/m_b^2)$ for \bar{m} . Values not given for the $\overline{\text{MS}}$ c quark mass, as the higher order terms are as large as the leading ones.

It is to be noted that pure QCD quarkonium determinations of m_b have remained remarkably stable since the earlier evaluations (ref. 5). [The variation of the *published* values is due to the variation of the favoured central value of Λ , from 200 to 283 MeV]. Also, the results depend very little on the value of μ chosen. Thus, even taking $\mu = m \simeq 5 \text{ GeV}$ one gets

$$m_b = 4901 \pm 60 \text{ MeV},$$

reasonably close to (2.6a), where we had elected $\mu = 2/a \simeq 2.88 \text{ GeV}$. The estimates of $\bar{m}_b(\bar{m}_b)$ are less stable, doubtlessly because of the large size of the $O(\alpha_s^3)$ corrections; but it is interesting

that this more precise spectroscopic determination of $\overline{\text{MS}}$ mass of the b quark mass given here agrees with recent determinations, based on sum rules, also accurate to $O(\alpha_s^3)$. These give^[15]

$$\bar{m}_b(\bar{m}_b) = 4260 \pm 100 \text{ MeV}.$$

We will discuss $\overline{\text{MS}}$ masses further later on.

The hyperfine splitting can likewise be evaluated, getting a prediction for the η_b mass:

$$M(\Upsilon) - M(\eta_b) = 53.3 \pm 12.5 \text{ MeV}.$$

The methods discussed for quarkonium can be extended with some work to hybrid states like $b\bar{c}$. The advantage here is that, since the masses of b, c quarks can be expressed in terms of those of the $\Upsilon, J/\psi$ particles, the theoretical errors are smaller than what one would get for charmonium. A recent $O(\alpha_s^4)$ calculation^[16] gives

$$M_{10}(b\bar{c}) = 6323 \pm 10 \pm 20 \text{ MeV},$$

the first error being perturbative, the second coming from estimated nonperturbative effects.

3. POWER CORRECTIONS. THE NONPERTURBATIVE VACUUM; RENORMALONS; SATURATION

As stated above, a calculation such as that in (2.6) includes leading nonperturbative effects. These are obtained by realizing that the motion of the $\bar{q}q$ pair takes place in the physical vacuum, chock full of soft gluons and light quark pairs so that, in particular, we have a nonzero value for the gluon condensate:

$$\langle \text{vac} | : \alpha_s G^2(0) : | \text{vac} \rangle \equiv \langle \alpha_s G^2 \rangle \neq 0.$$

The effects of this were first considered by Leutwyler and Voloshin^[17] (see also refs. 18, 5) and amount to a shift for (say) the ground state energy of quarkonium of

$$\delta_{\text{NP}}^{\langle \alpha_s G^2 \rangle} = m \frac{\pi \epsilon_{10} \langle \alpha_s : G^2 : \rangle}{(m C_F \alpha_s)^4}, \tag{3.1}$$

$\epsilon_{10} \simeq 1.5$. This is zero to all orders of perturbation theory, because so is the gluon condensate. On dimensional grounds one expects $\langle \alpha_s G^2 \rangle \sim \Lambda^4$, so that (3.1) is of order $\Lambda^4/m^3 \alpha_s^4$.

This is not the only power correction that may appear. We may have *renormalons*. Let us consider a one-gluon exchange diagram, for $\bar{q}q$ scattering. If we dress the gluon propagator with loops

then the corresponding potential, in momentum space, is

$$\tilde{V}(k) = \frac{-4\pi C_F}{k^2} \frac{4\pi}{\beta_0 \log(k^2/\Lambda^2)}, \quad (3.2)$$

and we have substituted the one-loop expression for $\alpha_s(k^2)$. (3.2) is undefined for *soft* gluons, with $k^2 \simeq \Lambda^2$. As follows from the general theory of singular functions, the ambiguity is of the form $c\delta(k^2 - \Lambda^2)$: upon Fourier transformation this produces an ambiguity in the x -space potential of $\delta V(r) = c[\sin \Lambda r]/r$. At short distances we expand this in powers of r and find^[19,20]

$$\delta_{\text{renorm.}} V(r) \sim C_0 + C_1 r^2 + \dots \quad (3.3)$$

The same result may be obtained with the more traditional method of Borel transforms. This coincides with the short distance behaviour of the nonperturbative potential as determined by Dosch and Simonov^[18].

The situation just described applies for states $\bar{q}q$ at short distances; but not so short that zero frequency gluons cannot separate the pair. If this last is the case, soft gluons do not resolve the $\bar{q}q$ pair and only see a dipole. The generated renormalon may then be seen (ref. 20) to correspond to the contribution of the gluon condensate in the Leutwyler-Voloshin mechanism.

What happens to the Aglietti and Ligeti renormalon? In fact, we have other long-distance power corrections. It is clear that the pole mass is defined purely in perturbation theory, and indeed one can check that a renormalon ambiguity appears already at one loop^[21]. Likewise, the coulombic potential is defined so that it vanishes at infinity: but, for confined quarks, “infinity” is equivalent to the confinement radius, $R \sim 1/\Lambda$. Actually, mass and potential renormalons cancel for the constant and quadratic terms in (3.2), in the very short distance regime; this has been verified by a detailed calculation in ref. 22, thus substantiating the intuitive arguments of ref. 20, already mentioned.

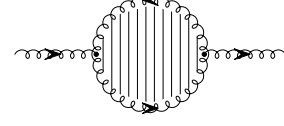


FIGURE 1. “Filled in” gluon loop.

Renormalons and the mechanism of Leutwyler–Voloshin are not the only possible sources of power corrections; we next consider *saturation*. We note that the ambiguities we have found are associated with small momenta or, equivalently, long distances. However, the *singularities* are clearly spurious. Indeed, not only the theory should be well defined but, because of confinement, long distances are never attained: the theory possesses an internal infrared cut-off of the order of the confinement radius, $R \sim \Lambda^{-1}$. To try and implement it we consider the gluon propagator. To one loop it gets a correction involving the vacuum polarization tensor. Neglecting quarks this is, in x -space, given by an expression like

$$\begin{aligned} \Pi_{\alpha\beta}^{aa'}(x, 0) &\sim g^2 f_{abc} f_{a'de} \langle 0 | \int d^4 y_1 d^4 y_2 \\ &\text{TB}_b^\alpha(y_1) \partial_\mu B_{c\alpha}(y_1) B_d^\beta(y_2) \partial_\nu B_{e\beta}(y_2) | 0 \rangle \\ &+ \dots \end{aligned}$$

We can take into account the *long distance* interactions by introducing a string between the field products at finite distances. In matrix notation for the gluonic fields, $\mathcal{B}^\mu = t_a B_a^\mu$, this is incorporated by replacing

$$\begin{aligned} &\mathcal{B}^\alpha(y_1) \mathcal{B}^\beta(y_2) \\ &\rightarrow \mathcal{B}^\alpha(y_1) \text{P} \left(\exp i \int_{y_2}^{y_1} dz^\mu \mathcal{B}_\mu(z) \right) \mathcal{B}^\beta(y_2). \end{aligned}$$

The process may be described as “filling the loop” (see Fig. 1) by introducing all exchanges between the gluonic lines there. If we furthermore replace the perturbative vacuum $|0\rangle$ by the non-perturbative one $|\text{vac}\rangle$, then we get a dressed propagator

$$D_{\text{dressed}}^{\mu\nu}(k) = D^{(0)\mu\nu}(k) \frac{4\pi}{\beta_0 \log(M^2 + k^2)/\Lambda^2},$$

and M^2 is related to the gluon condensate at finite

distances, $\langle G(x)G(0) \rangle_{\text{vac}}$. (for more details and references, see e.g. Simonov's lectures^[23]).

This suggests a *saturation* property of the coupling constant at small momenta (long distances) so that the expression for the running coupling constant should be modified according to

$$\alpha_s(k^2) = \frac{4\pi}{\beta_0 \log k^2/\Lambda^2} \quad (3.4)$$

$$\rightarrow \alpha_s^{\text{sat}}(k^2) = \frac{4\pi}{\beta_0 \log(k^2 + M^2)/\Lambda^2}.$$

It is certain that an expression such as (3.4) incorporates, to some extent, long distance properties of the QCD interaction. For example, if we take (3.4) with $M = \Lambda$ in the tree level potential for heavy quarks, this becomes the *Richardson potential*^[23]

$$V^{(0)}(\mathbf{k}) = - \frac{4\pi C_F \alpha_s(\mathbf{k}^2)}{\mathbf{k}^2}$$

$$\rightarrow V^{(0),\text{sat}}(\mathbf{k}) = - \frac{16\pi^2 C_F}{\beta_0 \mathbf{k}^2 \log(k^2 + \Lambda^2)/\Lambda^2}.$$

When one has $\mathbf{k}^2 \gg \Lambda^2$, the short distance coulombic potential is, of course, recovered. For $\mathbf{k}^2 \ll \Lambda^2$, however,

$$V^{(0),\text{sat}}(\mathbf{k}) \underset{\mathbf{k}^2 \ll \Lambda^2}{\simeq} \frac{16\pi^2 C_F \Lambda^2}{\beta_0 \mathbf{k}^4},$$

whose Fourier transform gives

$$V^{(0),\text{sat}}(r) \underset{r \gg \Lambda^{-1}}{\simeq} (\text{constant}) \times r,$$

i.e., a linear potential. Indeed, a reasonably accurate description of spin-independent splittings in quarkonia states is obtained with such a potential. However, this long-distance linear potential induced by saturation in the Richardson model is the *fourth component* of a Lorentz vector, while we know that the Wilson linear potential, as obtained, e. g., in the stochastic vacuum model or in lattice calculations, should be a Lorentz scalar. This is also obvious on phenomenological grounds as this potential has to provide attraction for *quark-quark* states in e.g. baryons. It thus follows that the linear potential obtained from saturation can be only of limited phenomenological use.

Saturation also gives a linear correction to the *short* distance coulombic interaction. The possibility of such a correction has been discussed by

several people; see, for example refs.. 20, 23. Personally I am not impressed by these arguments. It is clear that the QCD perturbative series is *not* convergent; in the case of quarkonia, the coefficients are not even analytic as one has terms in $\log \alpha_s$ (cf. (2.5c) and Sect. 4 below). The results one finds are then dependent on how one sums and, moreover, it is not guaranteed that results valid for *large* orders of perturbation theory will hold in the real world, where one knows at most three nontrivial terms. It is the author's belief that only if the summation method is rooted on solid physics it is likely to represent an improvement; otherwise, estimates of nonperturbative effects become pure guesswork. In this respect, the method of taking into account the nonperturbative nature of the physical vacuum by considering the effect of nonzero values for the correlators stands some chance of being meaningful, as indeed phenomenological calculations seem to indicate.

4. FURTHER DISCUSSION OF THE $\overline{\text{MS}}$ MASS: A MATTER OF CONVERGENCE

Write, for a heavy quark,

$$\bar{m}(\bar{m}) \equiv m/\{1 + \delta_1 + \delta_2 + \delta_3 + \dots\}, \quad (4.1a)$$

$$\delta_n = C_n \alpha_s^n,$$

with the coefficients C_n given in refs. 14 (see also ref. 11). Numerically one has

$$\begin{aligned} \delta_1(b) &= 0.090, & \delta_1(c) &= 0.137, \\ \delta_2(b) &= 0.045, & \delta_2(c) &= 0.108, \\ \delta_3(b) &= 0.029; & \delta_3(c) &= 0.125. \end{aligned} \quad (4.2)$$

Thus, the series relating pole and $\overline{\text{MS}}$ masses is at the edge of the region of convergence for the *b* quark, and clearly diverges for the third order contribution for the *c* quark.

We can use (4.2) and the formula for $M(\Upsilon)$ in terms of the pole mass to express the former in terms of \bar{m}_b : One finds, for the perturbative contributions, neglecting m_c^2/m_b^2 , and with Λ as before,

$$M(\Upsilon) = 2\bar{m}_b(\bar{m}_b) \left\{ 1 + C_F \frac{\alpha_s(\bar{m})}{\pi} \right. \\ \left. + 7.559 \left(\frac{\alpha_s(\bar{m})}{\pi} \right)^2 + 43.502 \left(\frac{\alpha_s}{\pi} \right)^3 \right\}. \quad (4.3)$$

(We have taken $n_f = 4$). This does not look particularly convergent, and is certainly not an

improvement over the perturbative expression using the pole mass where one has, for the choice $\alpha_s(\mu = C_F m_b \alpha_s)$ and still neglecting the masses of quarks lighter than the b ,

$$M(\Upsilon) = 2m_b \left\{ 1 - 2.193 \left(\frac{\alpha_s}{\pi} \right)^2 - 24.725 \left(\frac{\alpha_s}{\pi} \right)^3 - 458.28 \left(\frac{\alpha_s}{\pi} \right)^4 + 897.93 [\log \alpha_s] \left(\frac{\alpha_s}{\pi} \right)^5 \right\}. \quad (4.4)$$

To order three, (4.4) is actually better than (4.3); thus, and at least to third order, it is unclear that there is a connection between the rate of convergence and the use of pole or $\overline{\text{MS}}$ mass. What is more, at least the size of the *known* power corrections (namely, those stemming from the gluon condensate) do not favour the expression in terms of the $\overline{\text{MS}}$ mass. Indeed, if we evaluate the gluon condensate corrections to the direct formula (4.3) for $M(\Upsilon)$ in terms of $\bar{m}_b(\bar{m}_b)$, it turns out that they are larger than what one would have for the expression in terms of the pole mass, (4.4) (~ 80 against ~ 9 MeV), because of the definition of the renormalization point.

5. DECAYS OF QUARKONIA

Besides energies of bound states, once one has an effective interaction it is possible to evaluate, at least in principle, decay rates such as $\Upsilon \rightarrow e^+ e^-$. To relative order α_s (NLO) this has been done in ref. 5 (see also ref. 9) using the method of variations and the evaluation of ref. 25 for the hard part, δ_{rad} . We give here the exact result of a Rayleigh–Schrödinger perturbative calculation. One finds,

$$\begin{aligned} \Gamma(\Upsilon \rightarrow e^+ e^-) &= \Gamma^{(0)} [1 + \delta_{\text{wf}} + \delta_{\text{rad}} + \delta_{\text{NP}}]^2, \\ \Gamma^{(0)} &= 2 \left[\frac{Q_b \alpha_{\text{QED}}}{M(V)} \right]^2 (m C_F \alpha_s(\mu^2))^3; \\ \delta_{\text{wf}} + \delta_{\text{rad}} &= \left\{ \frac{3\beta_0}{4} \left(\log \frac{a\mu}{2} - \gamma_E - \frac{\pi^2}{9} + \frac{2}{3} \right) + \frac{3c^{(1)}}{2} - 2C_F \right\} \frac{\alpha_s}{\pi}; \end{aligned} \quad (5.1)$$

$c^{(1)}$ given in (2.1) and the nonperturbative contribution is

$$\delta_{\text{NP}} = \frac{1}{2} \left[\frac{270\,459}{108\,800} + \frac{1838\,781}{2\,890\,000} \right] \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 \tilde{\alpha}_s^6}.$$

The corrections are very large. Because of this the calculation is unreliable, even for $\bar{b}b$, and fails completely for $\bar{c}c$. One might hope that this would be arranged by the $O(\alpha_s^2)$ corrections (NNLO); but this appears not to be the case. We have, first, “hard” corrections^[26], similar to δ_{rad} ; and corrections to the wave function, given in refs. 11,27 where we send for the (rather lengthy) explicit formulas.

Numerically, and with the width in keV,

$\Gamma(\Upsilon \rightarrow e^+ e^-)$	LO	NLO	NNLO
$\mu = m$	0.41	1.22	1.13
$\mu = 2/a$	0.73	0.55	0.80

The experimental figure is

$$\Gamma(\Upsilon \rightarrow e^+ e^-) = 1.32 \pm 0.04 \text{ keV}.$$

Clearly, the large NLO and NNLO perturbative corrections, both of similar size, and of the leading NP correction, make the theoretical result unstable, as the Table shows clearly.

For the (perhaps measurable) toponium (T) width we get, for $m_t = 175$ GeV and still in keV,

$\Gamma(T \rightarrow e^+ e^-)$	LO	NLO	NNLO
$\mu = m_t$	6.86	10.53	13.0
$\mu = 2/a$	10.24	10.91	13.5

The situation has improved with respect to what we had for bottomium, but there is still a noticeable dependence on the renormalization point, and on the order of perturbation theory considered. We conclude on an *estimate* of some 11 – 14 keV for the width.

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